AMPLITUDE OF SUPERHORIZON COSMOLOGICAL PERTURBATIONS

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We study the influence of reheating on super-horizon density perturbations and gravitational waves. We correct wrong claims[1] about the joining of perturbations at cosmological transitions and about the quantization of cosmological perturbations.

1   Introduction

The aim of these proceedings, based on Ref. 2, is to clear up the recent controversy on the relative contribution of density perturbations and gravitational waves to the CMBR today.

The background model is taken to be a spatially flat FLRW model. We restrict our considerations to density perturbations and gravitational waves. The most general line element reads:

$$\mathrm{d}s^2 = a^2(\eta)\{-(1+2\phi)\mathrm{d}\eta^2 + 2B_{|i}\mathrm{d}x^i\mathrm{d}\eta + [(1-2\psi)\gamma_{ij} + 2E_{|i|j} + h_{ij}^{\mathrm{TT}}\mathrm{d}x^i\mathrm{d}x^j\} \ . \tag{1}$$

In the synchronous gauge, without loss of generality, the same line element can be written as:

$$\mathrm{d}s^2 = a^2(\eta)\{-\mathrm{d}\eta^2 + [(1+hQ)\delta_{ij} + \frac{h_l}{k^2}Q_{,i,j} + h_{\mathrm{gw}}Q_{ij}]\mathrm{d}x^i\mathrm{d}x^j\}. \tag{2}$$

The scalar function Q satisfies the Helmholtz equation and Q_{ij} is a symmetric, transverse and traceless spherical harmonic. k is the comoving wave number. In the following we use this form of the line element to make contact with previous works [1]. It is convenient to define $\mathcal{H} \equiv a'/a$, which is related with the Hubble constant H by the relation $H = \mathcal{H}/a$. A prime denotes a derivative with respect to conformal time. The quantity γ is defined by the expression: $\gamma(\eta) \equiv 1 - \mathcal{H}'/\mathcal{H}^2$. For the de Sitter space-time γ vanishes.

Let us now consider the equations of motion for the perturbed metric. For density perturbations (in the case of a vanishing anisotropic pressure), all relevant quantities can be expressed in terms of the variable μ defined by: $\mu \equiv a(h' + \mathcal{H}\gamma h)/(\mathcal{H}\sqrt{\gamma})$, except in the de Sitter case, which must be treated separately. For gravitational waves, the relevant quantity is $\mu_{\mathrm{gw}} \equiv ah_{\mathrm{gw}}$.

Using the perturbed Einstein equations, one can show that both types of perturbations obey the same class of equation, i.e. the equation of a parametric oscillator:

$$\mu'' + [k^2 - U(\eta)]\mu = 0 , \tag{3}$$

with $U_{\mathrm{dp}} = (a\sqrt{\gamma})''/(a\sqrt{\gamma})$ and $U_{\mathrm{gw}} = a''/a$. But there is a fundamental difference: the presence of the factor $\sqrt{\gamma}$ in the effective potential of density perturbations.

Equation (3) is valid for any model. However, it is important to consider cases where exact analytical solutions can be found. This happens for power law inflation where the scale factor is given by $a(\eta) = l_0|\eta|^{1+\beta}, \beta \leq -2$. We will consider a model with three epochs in succession: inflation, radiation-dominated era, and matter-dominated era. For this simple model, the function $\gamma(\eta)$ is a constant during each epoch and is given by: $\gamma_{\mathrm{i}} = (2+\beta)/(1+\beta), \gamma_{\mathrm{r}} = 2, \gamma_{\mathrm{m}} = 3/2$. Therefore the constant factor $\sqrt{\gamma}$ drops out of $U_{\mathrm{dp}}(\eta)$, and the solutions for density perturbations and gravitational waves are given by the same Bessel functions:

$$\mu(\eta) = (k\eta)^{1/2}[A_1 J_{\beta+1/2}(k\eta) + A_2 J_{-(\beta+1/2)}(k\eta)]. \tag{4}$$

The aim is now to compute the amplitude of both types of perturbations during the matter-dominated era. In order to perform this calculation, two questions must be addressed:

1) The initial conditions must be fixed. This amounts to choose the coefficients A_1 and A_2. This will be done with the help of quantum mechanical considerations.

2) The way the solutions are matched between different epochs must be specified. This is no problem for gravitational waves since the effective potential is well-defined (although discontinuous). This is more tricky for density perturbations. γ is a Heaviside function. This means that the effective potential of the density perturbations is not defined in the sense of distributions at the different transitions. Recently, there was a controversy on this point [1,3]. One purpose of Ref. 2 and this paper is to clear up this question. This question arises because we consider simple models which allow analytical solutions. In reality, the transition is smooth and the effective potential is never ill-defined.

2   Determination of the initial conditions

Let us start with the first question. The normalization of the perturbed scalar field is fixed by the uncertainty principle of Quantum Mechanics. In the high frequency regime, this leads to:

$$\delta\hat{\varphi}(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{2/3}} \int \mathrm{d}\mathbf{k}\hat{\varphi}_1(\eta, \mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{\sqrt{\hbar c}}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}\mathbf{k}}{\sqrt{2k}}[c_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + c_{\mathbf{k}}^{\dagger}(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{5}$$

where the annihilation and creation operators satisfy the usual commutation relation. A hat indicates that we are now dealing with operators. The normalization of the scalar perturbations is fixed by the normalization of the perturbed scalar field since they are linked through Einstein's equations. In the high frequency limit, this link is expressed as:

$$\lim_{k\to+\infty} \hat{\mu}(\eta, \mathbf{k}) = -\sqrt{2\kappa}a\hat{\varphi}_1(\eta, \mathbf{k}), \tag{6}$$

where $\kappa \equiv 8\pi G$. For the model under considerations, it is easy to find the time dependence of the creation and annihilation operators. As a consequence all quantities are fixed uniquely, if at some initial time η_0 the scalar field is placed in the vacuum state $c_{\mathbf{k}}(\eta = \eta_0)|0\rangle = 0$. For gravitational waves the line of reasoning is similar. The result[2] is that the A's satisfy $|A_1^{(\mathrm{gw})}| = |A_1^{(\mathrm{dp})}| = |A_2^{(\mathrm{gw})}| = |A_2^{(\mathrm{dp})}| = (2\pi l_{\mathrm{Pl}})/(|\cos(\pi\beta)|\sqrt{k})$. We will show below that $|A_1^{(\mathrm{gw})}/A_1^{(\mathrm{dp})}| = 1$ turns out to be crucial to predict the relative contribution of both types of perturbations.

3 The junction conditions

Let us now turn to the second question, i.e. the junction conditions. We analyze the approach taken by Grishchuk[1] and compare it with the claims of Deruelle and Mukhanov[3]. Suppose that the spatial transition hypersurface Σ is defined by its normal n^μ. To join two space-time manifolds along Σ without a surface layer the induced spatial metric $h_{ij} \equiv g_{ij} + n_i n_j$ and the extrinsic curvature $K_{ij} \equiv (-1/2)\mathcal{L}_n h_{ij}$ should be continuous on Σ. In order to compute K_{ij} the system of coordinates (i.e. the gauge) and the vector n^μ (i.e. the surface of transition) have to be specified. Different choices for n^μ lead to inequivalent junction conditions. If the surface of transition is defined by $q(\eta, x^i) = cte$, it was shown by Deruelle and Mukhanov[3] that the matching conditions read:

$$[\psi + \mathcal{H}\frac{\delta q}{q_0'}]_\pm = 0, \qquad [E]_\pm = 0, \tag{7}$$

$$[\psi' + \mathcal{H}\phi + (\mathcal{H}' - \mathcal{H}^2)\frac{\delta q}{q_0'}]_\pm = 0, \qquad [B - E' + \frac{\delta q}{q_0'}]_\pm = 0. \tag{8}$$

This result was generalized for spatially curved backgrounds and for the other types of perturbations in Ref. 2. If the surface of transition is a surface of constant time, the matching conditions in the synchronous gauge are:

$$[h]_\pm = [h']_\pm = [h_l]_\pm = [h_l']_\pm = 0. \tag{9}$$

For a sharp transition (i.e. if γ is discontinuous) the two sets are not equivalent. Since it is believed that the transition occurs on a surface of constant energy which is not a surface of constant time, using the second set of joining conditions in this situation leads to an error. However, if the transition is smooth, i.e. $[p]_\pm = 0$, then the two sets are equivalent, see Ref. 2.

We use this result to study the regularisation scheme of Grishchuk[1]: Instead of matching inflation to the radiation epoch directly, Grishchuk introduced a smooth transition in between. Physically, this smooth transition represents the reheating of the Universe. It starts at $\eta = \eta_1 - \epsilon$ (end of inflation) and ends at $\eta = \eta_1 + \epsilon$ (beginning of radiation). Here, ϵ is small compared to η_1 because we assume that reheating is a fast process. In the limit ϵ goes to zero, we recover the sharp transition considered before and $\gamma(\eta)$ becomes a Heaviside function jumping from $(2 + \beta)/(1 + \beta)$ to 2. For a smooth transition, without taking all the details of the reheating process into account, we do not know how the scale factor (and therefore γ) evolves between $\eta_1 - \epsilon$ and $\eta_1 + \epsilon$. The idea of Ref. 1 was to assume that the function $\gamma(\eta)$ is given by:

$$\gamma(\eta) = \frac{4 + 3\beta}{2(1 + \beta)} + \frac{\beta}{2(1 + \beta)}\tanh\left(\frac{\eta - \eta_1}{s}\right), \tag{10}$$

where s is a parameter controlling the sharpness of the transition. This equation holds for inflation and reheating, i.e. for η between $-\infty$ and $\eta_1 + \epsilon$. Formula (10) leads to a reasonable equation of state $p/\rho = -1 + 2/3\gamma(\eta)$ and therefore gives a reasonable approximation to the real (exact) complicated function $\gamma(\eta)$ even if details of the reheating process cannot be taken into account in such a simple approach.

From the previous discussion, it is clear that $\gamma(\eta)$ is always continuous, even at $\eta = \eta_1 + \epsilon$ where the explicit joining was performed in Ref. 1[a]. This means that $[p]_\pm$ vanishes. In Ref. 3, Deruelle and Mukhanov criticized the calculations done in Ref. 1 by means of the smooth transition described before, arguing that the junction conditions were not taken into account properly. We have shown that for continuous pressure the two sets of matching conditions are

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[a] In this paper η_1 was used instead of $\eta_1 + \epsilon$ to denote the end of reheating. It is very important to distinguish these two events.

equivalent. Therefore the claim of Deruelle and Mukhanov is not appropriate. For a smooth transition, the matching conditions used by Grishchuk are perfectly justified since they coincide with the ones derived in Ref. 3. The argument of Deruelle and Mukhanov would be relevant if the transition was sharp and $\gamma(\eta)$ discontinuous at $\eta = \eta_1 + \epsilon$.

Even the simple form (10) is too complicated to allow a direct integration of the equation of motion for μ. Nevertheless, we can follow the evolution of μ through inflation and reheating. For $\eta < \eta_1 - \epsilon$, $\gamma(\eta)$ is a constant and the solution for μ is given by Eq. (4). The value of $\mu(\eta)$ just before reheating can be easily computed:

$$\mu(\eta_1 - \epsilon) \simeq \frac{A_1^{(\mathrm{dp})}}{2^{\beta+\frac{1}{2}}\Gamma(\beta+\frac{3}{2})}[k(\eta_1 - \epsilon)]^{\beta+1} \simeq \frac{A_1^{(\mathrm{dp})}}{2^{\beta+\frac{1}{2}}\Gamma(\beta+\frac{3}{2})}(k\eta_1)^{\beta+1} \ , \tag{11}$$

because $k\eta_1 \ll 1$ and $\epsilon \ll \eta_1$. Between $\eta_1 - \epsilon$ and $\eta_1 + \epsilon$ the function $\gamma(\eta)$ is no longer a constant and the solution (4) can no longer be used. In order to evolve μ through the reheating transition we use the superhorizon solution of Eq. (3), $\mu \sim a\sqrt{\gamma}$, to obtain

$$\mu(\eta_1 + \epsilon) \simeq \frac{\mu(\eta_1 - \epsilon)}{a(\eta_1 - \epsilon)\sqrt{\gamma(\eta_1 - \epsilon)}}a(\eta_1 + \epsilon)\sqrt{\gamma(\eta_1 + \epsilon)} \simeq \mu(\eta_1 - \epsilon)\sqrt{\frac{2}{\gamma_\mathrm{i}}} \ , \tag{12}$$

because $a(\eta_1 + \epsilon) \approx a(\eta_1 - \epsilon)$. This relation should be compared to Eq. (81) and to the relation $\mu|_{\eta_1-0} = \mu|_{\eta_1+0}$ below Eq. (48) of Ref. 1. From Eq. (12), it is clear that the ratio $\mu(\eta_1 + \epsilon)/\mu(\eta_1 - \epsilon)$ is not 1 but proportional to $1/\sqrt{\gamma_\mathrm{i}}$. This factor is huge when γ_i is close to 0 (de Sitter). Therefore the mistake in Ref. 1 was not the use of wrong junction conditions but the fact that $\mu(\eta)$ was not evolved correctly through the reheating transition: $\gamma(\eta_1 - \epsilon) \neq \gamma(\eta_1 + \epsilon)$ implies $\mu(\eta_1 - \epsilon) \neq \mu(\eta_1 + \epsilon)$.

Let us turn to the radiation-matter transition taking place at equality. In Ref. 1 the equality transition was treated as a sharp transition. The joining conditions (9) are not correct in general (i.e., for any residual gauge fixing). However, if one specifies the synchronous gauge in the matter dominated epoch to be the comoving one, then the joining conditions (9) are fine at the equality transition[3]. That is because the density contrast vanishes at the leading order [i.e., it is proportional to $(k\eta)^2$].

Using the previous results, it is interesting to calculate the ratio of the Bardeen potential $\Phi \equiv \phi + [a(B - E')]'/a$ and h_gw at superhorizon scales today. We obtain (see Ref. 2):

$$\frac{h_\mathrm{gw}}{\Phi}\bigg|_\mathrm{today} = \frac{10\sqrt{2}}{3}\frac{\mathrm{A}_1^{(\mathrm{gw})}}{\mathrm{A}_1^{(\mathrm{dp})}}\frac{\mu(\eta_1 - \epsilon)}{\mu(\eta_1 + \epsilon)} = \frac{10}{3}\sqrt{\gamma_\mathrm{i}} \ . \tag{13}$$

This equation illustrates the importance of the initial conditions and the importance of the evolution of density perturbations during reheating. For small values of γ_i the amplitude of scalar metric perturbations is larger than the amplitude of gravitational waves, which implies that the main contribution to the large scale fluctuations in the cosmic microwave background is due to density fluctuations.

References

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Titre en français: Amplitude des perturbations cosmologiques plus grandes que l'horizon.

Résumé en français:

Nous étudions l'influence du 'reheating' sur les perturbations de densité et les ondes gravitationnelles plus grandes que l'horizon. Nous corrigeons de fausses affirmations faites à propos du raccordement des perturbations lors des transitions cosmologiques et à propos de leur quantification.